

The boundary of the irreducible components for invariant subspace varieties

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Abstract: Given partitions α, β, γ , the short exact sequences

$$0 \longrightarrow N_\alpha \longrightarrow N_\beta \longrightarrow N_\gamma \longrightarrow 0$$

of nilpotent linear operators of Jordan types α, β, γ , respectively, define a constructible subset $\mathbb{V}_{\alpha, \gamma}^\beta$ of an affine variety.

Geometrically, the varieties $\mathbb{V}_{\alpha, \gamma}^\beta$ are of particular interest as they occur naturally and since they typically consist of several irreducible components. In fact, each Littlewood-Richardson tableau Γ of shape (α, β, γ) contributes one irreducible component $\overline{\mathbb{V}}_\Gamma$.

We consider the partial order $\Gamma \leq_{\text{bound}}^* \tilde{\Gamma}$ on LR-tableaux which is the transitive closure of the relation given by $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma \neq \emptyset$. In this paper we compare the boundary relation with partial orders given by algebraic, combinatorial and geometric conditions. It is known that in the case where the parts of α are at most two, all those partial orders are equivalent. We prove that those partial orders are also equivalent in the case where $\beta \setminus \gamma$ is a horizontal and vertical strip. Moreover, we discuss how the orders differ in general.

MSC 2010: Primary: 14L30, 16G20, Secondary: 47A15

Key words: degenerations, partial orders, nilpotent operators, invariant subspaces, Littlewood-Richardson tableaux

1. Introduction

Often in geometry, naturally occurring conditions define subsets of varieties which are either very big in size or tiny. For example, among all linear operators acting on a given finite dimensional vector space, the invertible ones

The first named author is partially supported by Research Grant No. DEC-2011/02/A/ ST1/00216 of the Polish National Science Center.

This research is partially supported by a Travel and Collaboration Grant from the Simons Foundation (Grant number 245848 to the second named author).

form an open and dense subset. And so do, among all nilpotent operators, those which have only one Jordan block. A notable exception to this rule occurs in the variety of short exact sequences of nilpotent linear operators; it is partitioned, by means of Littlewood-Richardson tableaux, into components of equal dimension. They are the topic of this paper.

Throughout we assume that k is an algebraically closed field. Each nilpotent k -linear operator is given uniquely, up to isomorphy, as a $k[T]$ -module $N_\alpha = \bigoplus_{i=1}^s k[T]/(T^{\alpha_i})$ for some partition $\alpha = (\alpha_1, \dots, \alpha_s)$ which represents the sizes of its Jordan blocks.

The Theorem of Green and Klein [8] states that for given partitions α, β, γ , there exists a short exact sequence

$$0 \longrightarrow N_\alpha \longrightarrow N_\beta \longrightarrow N_\gamma \longrightarrow 0$$

of nilpotent linear operators if and only if there is a Littlewood-Richardson (LR-) tableau of shape (α, β, γ) . The collection of all such short exact sequences forms a variety $\mathbb{V}_{\alpha, \gamma}^\beta(k)$ which can be partitioned using LR-tableaux, as follows. Consider the affine variety $\text{Hom}_k(N_\alpha, N_\beta)$ of k -linear maps endowed with the Zariski topology, and assume that all subsets carry the induced topology. Define

$$\mathbb{V}_{\alpha, \gamma}^\beta(k) = \{f : N_\alpha \rightarrow N_\beta \mid f \text{ monomorphism of } k[T]\text{-modules} \\ \text{with cokernel isomorphic to } N_\gamma\}.$$

The irreducible components of $\mathbb{V}_{\alpha, \gamma}^\beta(k)$ are counted by the Littlewood-Richardson coefficient. Namely, to each monomorphism in $\mathbb{V}_{\alpha, \gamma}^\beta$ one can associate an LR-tableau Γ of shape (α, β, γ) , as we will see in Section 2. The subset \mathbb{V}_Γ of $\text{Hom}_k(N_\alpha, N_\beta)$ of all such monomorphisms is constructible and irreducible. All the \mathbb{V}_Γ have the same dimension. We denote by $\overline{\mathbb{V}}_\Gamma$ the closure of \mathbb{V}_Γ in $\mathbb{V}_{\alpha, \gamma}^\beta$; the sets $\overline{\mathbb{V}}_\Gamma$ define the irreducible components of $\mathbb{V}_{\alpha, \gamma}^\beta$, they are indexed by the set $\mathcal{T}_{\alpha, \gamma}^\beta$ of all LR-tableaux of shape (α, β, γ) (see [15, Theorem 4.3] and [17]).

Our aim in this paper is to shed light on the geometry in the variety

$$\mathbb{V}_{\alpha, \gamma}^\beta = \bigcup_{\Gamma \in \mathcal{T}_{\alpha, \gamma}^\beta} \mathbb{V}_\Gamma;$$

by studying the boundary relation given as follows.

$$(1.1) \quad \Gamma \leq_{\text{bound}} \tilde{\Gamma} \Leftrightarrow \mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma} \neq \emptyset \quad \text{where} \quad \Gamma, \tilde{\Gamma} \in \mathcal{T}_{\alpha, \gamma}^{\beta}.$$

We are ready to present the main results of the paper.

1.1. Two combinatorial criteria

On the set $\mathcal{T}_{\alpha, \gamma}^{\beta}$, there are two partial orders of combinatorial nature. The *dominance relation* \leq_{dom} is given by the natural partial orders of the partitions defining the tableaux. The second relation is the *box-order*, it is given by repeatedly exchanging two entries in the tableau in such a way that the smaller entry moves up and such that the lattice permutation condition is preserved. We introduce the two orders formally in Section 2.1.

The following result presents a necessary and a sufficient criterion for two LR-tableaux to be in boundary relation (see Section 3.1 for a proof of (a); Sections 4 and 5 for a proof of (b)).

THEOREM 1.2. *Given partitions α, β, γ , the following implications hold for LR-tableaux $\Gamma, \tilde{\Gamma}$ of shape (α, β, γ) .*

(a) *If $\Gamma \leq_{\text{bound}} \tilde{\Gamma}$ then $\Gamma \leq_{\text{dom}} \tilde{\Gamma}$.*

(b) *Suppose $\beta \setminus \gamma$ is a horizontal strip. If $\Gamma \leq_{\text{box}} \tilde{\Gamma}$ then $\Gamma \leq_{\text{bound}}^* \tilde{\Gamma}$.*

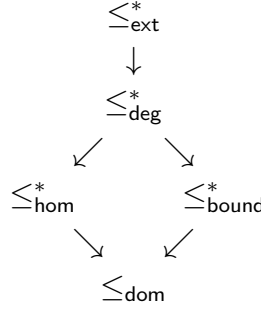
As a consequence of (a), the relation \leq_{bound} is antisymmetric. Hence, the transitive closure \leq_{bound}^* is a partial order on $\mathcal{T}_{\alpha, \gamma}^{\beta}$.

1.2. Two algebraic tests

The algebraic group $G = \text{Aut}_{k[T]}(N_{\alpha}) \times \text{Aut}_{k[T]}(N_{\beta})$ acts on $\mathbb{V}_{\alpha, \gamma}^{\beta}$ via $(a, b) \cdot f = bfa^{-1}$. The orbits of this group action are in one-to-one correspondence with the isomorphism classes of embeddings $f : N_{\alpha} \rightarrow N_{\beta}$.

We consider the following preorder for LR-tableaux. We say $\Gamma \leq_{\text{deg}} \tilde{\Gamma}$ if there are embeddings $f \in \mathbb{V}_{\Gamma}, \tilde{f} \in \mathbb{V}_{\tilde{\Gamma}}$ such that $f \leq_{\text{deg}} \tilde{f}$, that is, $\mathcal{O}_{\tilde{f}} \subset \overline{\mathcal{O}_f}$, where \mathcal{O}_f is the orbit of f under the action of G on $\mathbb{V}_{\alpha, \gamma}^{\beta}$. The degeneration relation is under control algebraically as the ext-relation implies the deg-relation, which in turn implies the hom-relation (see Section 4).

As the deg-relation, also the hom- and ext-relations give rise to preorders for LR-tableaux. In the diagram below, the relations introduced so far on the set $\mathcal{T}_{\alpha,\gamma}^\beta$ are ordered vertically by containment, with the dominance order the weakest of the relations pictured.



We show that the dominance order is in fact equivalent to the hom-order restricted to certain objects called *pickets*. Thus we have also algebraic tests both for the validity and for the failure of the boundary relation. We present proofs in Section 4.

THEOREM 1.3. *Suppose α, β, γ are partitions. The following implications hold for LR-tableaux $\Gamma, \tilde{\Gamma}$ of shape (α, β, γ) :*

$$\Gamma \leq_{\text{ext}} \tilde{\Gamma} \implies \Gamma \leq_{\text{bound}} \tilde{\Gamma} \implies \Gamma \leq_{\text{hom-picket}} \tilde{\Gamma}.$$

1.3. Horizontal and vertical strips

We can say more for tableaux with additional properties. If the partitions α, β, γ are such that $\beta \setminus \gamma$ is a horizontal strip, the box-order implies the ext-relation, as we will show in Section 5.

PROPOSITION 1.4. *Suppose $\Gamma, \tilde{\Gamma}$ are LR-tableaux which have the same shape and which are horizontal strips. If $\Gamma \leq_{\text{box}} \tilde{\Gamma}$ then $\Gamma \leq_{\text{ext}}^* \tilde{\Gamma}$.*

In the special case where the partitions are such that $\beta \setminus \gamma$ is a horizontal and vertical strip, the combinatorial relations \leq_{box} and \leq_{dom} are equivalent. In [13] we give two proofs for this statement; in Section 2.1 we sketch the algorithmic approach in one of them. We deduce the following result.

THEOREM 1.5. *Suppose α, β, γ are partitions such that $\beta \setminus \gamma$ is a horizontal and vertical strip. The following relations are partial orders which are equivalent to each other.*

$$\leq_{\text{box}}, \leq_{\text{ext}}^*, \leq_{\text{deg}}^*, \leq_{\text{hom}}^*, \leq_{\text{bound}}^*, \leq_{\text{dom}}.$$

Remark: In Section 3.2 we provide an example showing that the assumption that $\beta \setminus \gamma$ is a vertical strips is necessary in Theorem 1.5.

For comparison we recall the following result from [11, 12].

THEOREM 1.6. *Suppose α, β, γ are partitions such that all parts of α are at most two. The relations $\leq_{\text{dom}}, \leq_{\text{hom}}, \leq_{\text{bound}}, \leq_{\text{deg}}, \leq_{\text{ext}}, \leq_{\text{box}}$ are all partial orders which are equivalent to each other.*

Related results. The Theorem of Gerstenhaber and Hesselink shows that the natural partial order of partitions is equivalent to the degeneration order of nilpotent linear operators, see [5, 6, 14]. We investigate a similar problem: connections of the dominance order of LR-tableaux with the boundary order defined below. Also extensions of nilpotent linear operators are of interest as they are connected with the classical Hall algebras and Hall polynomials, see [16]. Well understood are generic extensions and their relationships with the specializations to $q = 0$ of the Ringel-Hall algebras, see [3, 4, 10, 18, 19].

Organization of this paper. In Section 2, we describe how partitions and tableaux describe short exact sequences of linear operators, or equivalently of *embeddings* or *invariant subspaces* of linear operators. We introduce special types of embeddings, namely pickets and poles, and show that every LR-tableau which is a horizontal strip can be realized by a direct sum of poles and empty pickets.

In Section 3, we show that the boundary relation in Formula 1.1 is a preorder and present an example showing that \leq_{bound} may not be transitive. Moreover, we prove that the boundary relation implies the dom-order (Part (a) of Theorem 1.2).

In Section 4, we adapt the ext- deg- and hom-relations for modules to tableaux. As for modules, the ext-order implies the degeneration order, which implies the hom-order. Moreover, the hom-relation implies the dominance order. This completes the proof of Theorem 1.3.

In Section 5, we consider LR-tableaux which are horizontal strips and show that the box-order implies the ext-relation (Proposition 1.4), and hence the

boundary relation (Part (b) of Theorem 1.2). We complete the proof of Theorem 1.5.

Acknowledgements. The authors would like to thank Hugh Thomas for discussions which led to the proof of Theorem 2.2, and for contributing the example in Section 3.2.

This research project was started when the authors visited the Mathematisches Forschungsinstitut Oberwolfach in a Research in Pairs project. They would like to thank the members of the Forschungsinstitut for creating an outstanding environment for their mathematical research.

2. Littlewood-Richardson tableaux

Given three partitions, α, β, γ , we consider the set $\mathcal{T}_{\alpha, \gamma}^{\beta}$ of all Littlewood-Richardson tableaux of shape (α, β, γ) . We define the domination order on the set $\mathcal{T}_{\alpha, \gamma}^{\beta}$. Moreover, we introduce the LR-tableau of a short exact sequence, and determine the tableaux for certain types of short exact sequences, namely pickets and poles. For the case where the skew diagram $\beta \setminus \gamma$ is a horizontal strip, we introduce the box-order and show that any LR-tableau of shape (α, β, γ) can be realized as the LR-tableau of a direct sum of poles and empty pickets.

2.1. Combinatorial orders on the set of LR-tableaux

Notation: Recall that a *partition* $\alpha = (\alpha_1, \dots, \alpha_s)$ is a finite non-increasing sequence of natural numbers; we picture α by its Young diagram which consists of s columns of length given by the parts of α . The *transpose* α' of α is given by the formula

$$\alpha'_j = \#\{i : \alpha_i \geq j\},$$

it is pictured by the transpose of the Young diagram for α . Two partitions $\alpha, \tilde{\alpha}$ are in the *natural partial order*, in symbols $\alpha \leq_{\text{nat}} \tilde{\alpha}$, if the inequality

$$\alpha'_1 + \dots + \alpha'_j \leq \tilde{\alpha}'_1 + \dots + \tilde{\alpha}'_j$$

holds for each j .

Given three partitions α, β, γ , an *LR-tableau* of shape (α, β, γ) is a Young diagram of shape β in which the region $\beta \setminus \gamma$ contains α'_1 entries $\boxed{1}$, \dots , α'_s entries \boxed{s} , where $s = \alpha_1$ is the largest entry, such that

- in each row, the entries are weakly increasing,
- in each column, the entries are strictly increasing,
- for each $\ell > 1$ and for each column c : on the right hand side of c , the number of entries $\ell - 1$ is at least the number of entries ℓ .

The skew diagram $\beta \setminus \gamma$ is said to be a *horizontal strip* if $\beta_i \leq \gamma_i + 1$ holds for all i , and a *vertical strip* if $\beta' \setminus \gamma'$ is a horizontal strip.

Example: Let $\alpha = (3, 2)$, $\beta = (4, 3, 3, 2, 1)$, $\gamma = (3, 2, 2, 1)$. Then the transpose of α is $\alpha' = (2, 2, 1)$, so we have to fill the skew diagram $\beta \setminus \gamma$ with two $\boxed{1}$'s, two $\boxed{2}$'s, and one $\boxed{3}$. Due to the conditions on an LR-tableau, this can be done in exactly two ways.



In this example, $\beta \setminus \gamma$ is a horizontal but not a vertical strip.

Notation: One can represent an LR-tableau Γ by a sequence of partitions

$$\Gamma = [\gamma^{(0)}, \dots, \gamma^{(s)}]$$

where $\gamma^{(i)}$ denotes the region in the Young diagram β which contains the entries $\boxed{1}, \boxed{2}, \dots, \boxed{i}$. If Γ has shape (α, β, γ) , then $\gamma = \gamma^{(0)}$, $\beta = \gamma^{(s)}$, and $\alpha'_i = |\gamma^{(i)} \setminus \gamma^{(i-1)}|$ for $i = 1, \dots, s$.

In the example above, the first tableau is given by the sequence of partitions $\Gamma = [(3, 2, 2, 1), (3, 3, 2, 1, 1), (4, 3, 2, 2, 1), (4, 3, 3, 2, 1)]$.

We introduce two partial orders on the set $\mathcal{T}_{\alpha, \gamma}^{\beta}$ of all LR-tableaux of shape (α, β, γ) .

Definition: Two LR-tableaux $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(s)}]$, $\tilde{\Gamma} = [\tilde{\gamma}^{(0)}, \dots, \tilde{\gamma}^{(s)}]$ of the same shape are in the *dominance order*, in symbols $\Gamma \leq_{\text{dom}} \tilde{\Gamma}$, if for each i , the corresponding partitions $\gamma^{(i)}$, $\tilde{\gamma}^{(i)}$ are in the natural partial order, i.e. $\gamma^{(i)} \leq_{\text{nat}} \tilde{\gamma}^{(i)}$.

Definition: Suppose $\Gamma, \tilde{\Gamma}$ are LR-tableaux of the same shape which we assume to be a horizontal strip. We say $\tilde{\Gamma}$ is obtained from Γ by a *box move* if after two entries in Γ have been exchanged in such a way that the smaller entry is in the higher position in $\tilde{\Gamma}$, we obtain $\tilde{\Gamma}$ by re-sorting the list of columns if necessary. We denote by \leq_{box} the partial order generated by box moves.

Here is an example:

$$\Gamma : \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & \\ \hline & 3 & & \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline \end{array} <_{\text{box}} \tilde{\Gamma} : \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 1 & \\ \hline & 2 & & \\ \hline & 2 & & \\ \hline 3 & & & \\ \hline \end{array}$$

LEMMA 2.1. *For LR-tableaux of the same shape, the \leq_{box} -order implies the \leq_{dom} -order.*

Proof. Suppose the LR-tableau $\tilde{\Gamma} = [\tilde{\gamma}^{(0)}, \dots, \tilde{\gamma}^{(s)}]$ is obtained from $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(s)}]$ by a box move based on entries i and j with, say, $i < j$. The process of reordering the entries in each row will not affect entries less than i or larger than j , so the partitions $\gamma^{(0)}, \dots, \gamma^{(i-1)}$, and $\gamma^{(j)}, \dots, \gamma^{(s)}$ remain unchanged. The partitions $\gamma^{(\ell)}, \tilde{\gamma}^{(\ell)}$ for $i \leq \ell < j$ are different and satisfy $\gamma^{(\ell)} <_{\text{nat}} \tilde{\gamma}^{(\ell)}$ (since the defining partial sums can only increase). This shows that $\Gamma <_{\text{dom}} \tilde{\Gamma}$. \square

The converse does not always hold, not even for horizontal strips:

Example: Let $\beta = (4, 3, 3, 2, 1)$, $\gamma = (3, 2, 2, 1)$ and $\alpha = (3, 2)$. We have seen that there are two LR-tableaux of type (α, β, γ) . They are incomparable in \leq_{box} -relation, but

$$\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & \\ \hline 1 & 3 & & \\ \hline 2 & & & \\ \hline \end{array} \leq_{\text{dom}} \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 1 & \\ \hline & 2 & 2 & \\ \hline 3 & & & \\ \hline \end{array}$$

However for horizontal and vertical strips, the two partial orders are equivalent [13]:

THEOREM 2.2. *Suppose α, β, γ are partitions such that $\beta \setminus \gamma$ is a horizontal and vertical strip. Then the two partial orders $\leq_{\text{dom}}, \leq_{\text{box}}$ are equivalent on $\mathcal{T}_{\alpha, \gamma}^{\beta}$.*

In [13] we present two proofs of the fact that \leq_{dom} implies \leq_{box} (for horizontal and vertical strips). One of them is algorithmic. Below we present this algorithm without any proof of its correctness. The reader is referred to [13] for details and proofs.

Algorithm: For an LR-tableau Γ we denote by $\omega(\Gamma)$ the list of entries when read from left to right, and each column from the bottom up. Clearly, Γ is determined uniquely by its shape and by the list of entries.

Input: LR-tableaux $\Gamma, \tilde{\Gamma}$ of the shape (α, β, γ) such that $\beta \setminus \gamma$ is a vertical and a horizontal strip and $\Gamma <_{\text{dom}} \tilde{\Gamma}$.

Output: LR-tableau $\widehat{\Gamma}$ of the shape (α, β, γ) such that $\Gamma \leq_{\text{dom}} \widehat{\Gamma}$ and $\widehat{\Gamma} <_{\text{box}} \widetilde{\Gamma}$.

Step 1. Find the smallest k such that $\omega(\Gamma)_k \neq \omega(\widetilde{\Gamma})_k$ and put $x = \omega(\Gamma)_k$.

Step 2. Choose minimal $m \geq k + 1$ such that $x = \omega(\widetilde{\Gamma})_m$.

Step 3. Let $y = \min\{\omega(\widetilde{\Gamma})_i > x : k \leq i < m\}$.

Step 4. Choose $k \leq l < m$ such that $y = \omega(\widetilde{\Gamma})_l$.

Step 5. Define $\widehat{\Gamma}$ such that $\omega(\widehat{\Gamma})_i = \omega(\widetilde{\Gamma})_i$, for $i \neq l, m$, and $\omega(\widehat{\Gamma})_l = x$, $\omega(\widehat{\Gamma})_m = y$.

Example: Let $\beta = (6, 5, 4, 3, 2, 1)$, $\gamma = (5, 4, 3, 2, 1)$ and $\alpha = (3, 2, 1)$. Consider two LR-tableaux Γ and $\widetilde{\Gamma}$ of the shape (α, β, γ) such that $\omega(\Gamma) = (1, 3, 2, 2, 1, 1)$ and $\omega(\widetilde{\Gamma}) = (2, 3, 2, 1, 1, 1)$. It is straightforward to check that $\beta \setminus \gamma$ is a horizontal and vertical strip and $\Gamma <_{\text{dom}} \widetilde{\Gamma}$.

$$\Gamma : \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline & & & & 1 & \\ \hline & & & 2 & & \\ \hline & & 2 & & & \\ \hline & 3 & & & & \\ \hline 1 & & & & & \\ \hline \end{array} <_{\text{dom}} \widetilde{\Gamma} : \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 1 \\ \hline & & & & 1 & \\ \hline & & & 1 & & \\ \hline & & 2 & & & \\ \hline & 3 & & & & \\ \hline 2 & & & & & \\ \hline \end{array}$$

We apply the algorithm. Note that $k = 1$, $x = 1$ and $m = 4$. Now we can choose $y = \omega(\widetilde{\Gamma})_1 = 2$ or $y = \omega(\widetilde{\Gamma})_3 = 2$. If we choose $y = \omega(\widetilde{\Gamma})_1$, then $\widehat{\Gamma} = \Gamma$. In the second case, i.e. if $y = \omega(\widetilde{\Gamma})_3$, we get $\omega(\widehat{\Gamma}) = (2, 3, 1, 2, 1, 1)$. It is easy to see that $\Gamma <_{\text{dom}} \widehat{\Gamma}$ and we can continue.

2.2. The LR-tableau of a short exact sequence

Notation: By a *nilpotent operator* we understand a pair (V, T) where V is a finite dimensional k -vector space and $T : V \rightarrow V$ a k -linear nilpotent operator. Each such pair is determined uniquely, up to isomorphy, by the partition $\alpha = (\alpha_1, \dots, \alpha_s)$ which records the sizes of the Jordan blocks. We consider (V, T) as the module over the polynomial ring

$$N_\alpha := \bigoplus_{i=1}^s k[T]/(T^{\alpha_i}).$$

Conversely, given a $k[T]$ -module M on which the variable T acts nilpotently, the transpose of the partition β such that $M \cong N_\beta$ is given by $\beta'_\ell = \dim \frac{T^{\ell-1}M}{T^\ell M}$.

Given three partitions α, β, γ , there is a short exact sequence $E : 0 \rightarrow N_\alpha \rightarrow N_\beta \rightarrow N_\gamma \rightarrow 0$ if and only if there is an LR-tableau of shape (α, β, γ) [8]. The *tableau* Γ *corresponding to the sequence* E is obtained as follows. Let B be the $k[T]$ -module N_β and A the submodule given by the image of the monomorphism $N_\alpha \rightarrow N_\beta$. The partitions defining $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(s)}]$ are obtained as the isomorphism types of the nilpotent operators [16, II, (1.4)]:

$$N_{\gamma^{(i)}} = B/T^i A.$$

Definition: Given two partitions $\gamma, \tilde{\gamma}$, the *union* $\gamma \cup \tilde{\gamma}$ has as Young diagram the sorted union of the columns in the Young diagrams for γ and $\tilde{\gamma}$, in symbols, $(\gamma \cup \tilde{\gamma})'_i = \gamma'_i + \tilde{\gamma}'_i$.

For two tableaux $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(s)}]$, $\tilde{\Gamma} = [\tilde{\gamma}^{(0)}, \dots, \tilde{\gamma}^{(t)}]$, the *union* of the tableaux is given rowwise:

$$\Gamma \cup \tilde{\Gamma} = [\gamma^{(0)} \cup \tilde{\gamma}^{(0)}, \dots, \gamma^{(m)} \cup \tilde{\gamma}^{(m)}]$$

where $m = \max\{s, t\}$ and $\gamma^{(i)} = \gamma^{(s)}$ for $i > s$ and $\tilde{\gamma}^{(i)} = \tilde{\gamma}^{(t)}$ for $i > t$.

LEMMA 2.3. *Suppose the exact sequences E, \tilde{E} have LR-tableaux $\Gamma, \tilde{\Gamma}$, respectively. Then the LR-tableau of the direct sum $E \oplus \tilde{E}$ is $\Gamma \cup \tilde{\Gamma}$.*

Proof. Suppose E, \tilde{E} are given by the embeddings $A \subset B, \tilde{A} \subset \tilde{B}$. The j -th partition in the LR-tableau for $E \oplus \tilde{E}$ is the Jordan type for $B/T^j A \oplus \tilde{B}/T^j \tilde{A}$, which is $\gamma^{(j)} \cup \tilde{\gamma}^{(j)}$. \square

Thus, the LR-tableau of a direct sum is obtained by merging the rows of the LR-tableaux of the summands, starting at the top, and by sorting the entries in each row.

We present a formula for the number $\mu_{\ell, r}$ of boxes \square in the r -th row in the LR-tableau $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(t)}]$ of an embedding $(A \subset B)$. We refer to [22, Theorem 1] for a module-theoretic and homological interpretation of this number.

Denote by $\gamma_{\leq r} = (\gamma'_1, \dots, \gamma'_r)'$ the partition which consists of the first r rows of γ . Thus, if a $k[T]$ -module C has type γ , then $C/T^r C$ has type $\gamma_{\leq r}$. In particular, the first r rows of the partitions $\gamma^{(\ell)}$ are given as follows.

$$(2.4) \quad \gamma_{\leq r}^{(\ell)} = \text{type } \frac{B}{T^\ell A + T^r B}$$

As an immediate consequence, the number of boxes $\boxed{\ell}$ in the first r rows of Γ is given by

$$|\gamma_{\leq r}^{(\ell)} \setminus \gamma_{\leq r}^{(\ell-1)}| = \dim \frac{T^{\ell-1}A + T^r B}{T^\ell A + T^r B},$$

and the formula for $\mu_{\ell,r}$ is as follows.

$$\begin{aligned} (2.5) \quad \mu_{\ell,r}(A \subset B) &= |\gamma_{\leq r}^{(\ell)} \setminus \gamma_{\leq r}^{(\ell-1)}| - |\gamma_{\leq r-1}^{(\ell)} \setminus \gamma_{\leq r-1}^{(\ell-1)}| \\ &= \dim \frac{T^{\ell-1}A + T^r B}{T^\ell A + T^r B} - \dim \frac{T^{\ell-1}A + T^{r-1}B}{T^\ell A + T^{r-1}B} \end{aligned}$$

In the remainder of this section we study several types of examples.

2.3. Example 1: Pickets

Definition: A short exact sequence $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a *picket* if B is indecomposable as a $k[T]$ -module (so the partition β has only one part). A picket E is *empty* if $A = 0$.

Remark: Recall that the invariant subspaces of a linear operator with only one Jordan block are determined uniquely by their dimension. As a consequence, a picket E as above is determined uniquely, up to isomorphism, by the dimensions $n = \dim B$ and $m = \dim A$. We write

$$P_m^n := (0 \rightarrow (T^{n-m}) \subset k[T]/(T^n) \rightarrow k[T]/(T^{n-m}) \rightarrow 0).$$

We picture pickets as follows. In the diagram, the column represents the Jordan block of B and the dot in the $(n - m + 1)$ -st box the submodule generator T^{n-m} in B .

$$P_2^5 : \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \bullet \\ \hline \square \\ \hline \end{array} \qquad \Gamma : \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \boxed{1} \\ \hline \boxed{2} \\ \hline \end{array}$$

To determine the LR-tableau $\Gamma = [\gamma^{(0)}, \dots, \gamma^{(t)}]$ of a picket, note that $t = m$, $\gamma^{(0)} = \text{type } B/A = (n - m)$, $\gamma^{(1)} = \text{type } B/TA = (n - m + 1)$, \dots , $\gamma^{(m)} = \text{type } B = (n)$.

2.4. Example 2: Poles

Definition: A short exact sequence $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a *pole* if A is indecomposable as a $k[T]$ -module and E is indecomposable as a short exact sequence.

For example, every non-empty picket is a pole. Poles have been classified, up to isomorphism, by Kaplansky [7, Theorem 24].

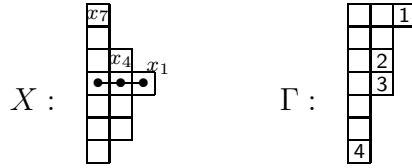
THEOREM 2.6. *A pole with submodule generator a is determined uniquely, up to isomorphism, by the radical layers of the elements $T^i a$.* \square

We denote by $P(x_1, \dots, x_k)$ the pole given by the radical layers x_1, \dots, x_k . Thus if a is the submodule generator, then $T^i a \in T^{x_i} B \setminus T^{x_i+1} B$. For example for $1 \leq \ell \leq m$, the picket P_ℓ^m is the pole $P(m - \ell, m - \ell + 1, \dots, m - 1)$.

LEMMA 2.7. *In the LR-tableau of a pole $P(x_1, \dots, x_k)$, the entries $1, \dots, k$ occur in rows $x_1 + 1, \dots, x_k + 1$, respectively.*

Proof. Consider the pole P as an embedding $(A \subset B)$ and apply Formula (2.5). The number $\mu_{\ell,r}$ is either 0 or 1 since A is cyclic. It takes value 1 if $\ell = i$ for some $1 \leq i \leq k$ and $r = x_i + 1$. \square

Example: The pole $X = P(0, 2, 3, 6)$ has the LR-tableau as pictured, in particular, $\alpha = (4)$, $\beta = (741)$, $\gamma = (62)$. We visualize the embedding $(A \subset B)$ where $A \cong N_\alpha$, $B \cong N_\beta$ using the conventions from [21, (2.3)]. If the generators of B as a $k[T]$ -module are x_7, x_4, x_1 , as indicated, then the submodule generator $a = T^3 x_7 + T x_4 + x_1$ is given by the row of connected bullets.



To verify, we compute the sequence of radical layers given by the T -powers $T^i a$ of the submodule generator a , and the corresponding quotients $B/(T^i a)$.

$$\begin{array}{llll}
 a = & T^3 x_7 + T x_4 + x_1 & \in B \setminus TB & B/(a) \cong N_{62} \\
 Ta = & T^4 x_7 + T^2 x_4 & \in T^2 B \setminus T^3 B & B/(Ta) \cong N_{621} \\
 T^2 a = & T^5 x_7 + T^3 x_4 & \in T^3 B \setminus T^4 B & B/(T^2 a) \cong N_{631} \\
 T^3 a = & T^6 x_7 & \in T^6 B \setminus 0 & B/(T^3 a) \cong N_{641}
 \end{array}$$

Hence the LR-tableau of the embedding is $\Gamma = [62, 621, 631, 641, 741]$.

Definition: A short exact sequence $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an *extended pole* if A is indecomposable as a $k[T]$ -module.

Clearly, each extended pole is a direct sum of a pole and a sum of empty pickets. Using the above results, they can be characterized as follows.

LEMMA 2.8. *The isomorphism type of an extended pole $(A \subset B)$ where $A = (a)$ is determined uniquely by each of the following:*

1. *the type of B and the radical layers of the elements $T^i a$,*
2. *the LR-tableau for $(A \subset B)$, or*
3. *the type of B and the type of B/A .*

Proof. The extended pole is the direct sum of a pole and a sum of empty pickets. The radical layers of the elements $T^i a$, which can be read off from the positions of the entries in the LR-tableau (Lemma 2.7), determine the pole uniquely, up to isomorphism (Theorem 2.6). The ambient space of the pole is a direct summand of B . The isomorphism type of a direct complement determines the empty pickets. \square

2.5. Horizontal strips

We consider short exact sequences for which the LR-tableau is a horizontal strip. Such a sequence can be decomposed as a direct sum of extended poles. Moreover, the decomposition can be chosen such that it is compatible with a given box move.

LEMMA 2.9. *Suppose Γ is an LR-tableau and a horizontal strip which is not empty. Then $\Gamma = \Gamma^e \cup \Gamma'$ where:*

- *Γ^e is the LR-tableau of an extended pole (i.e., there are no multiple entries), it is a horizontal strip and has no empty columns;*
- *and Γ' is an LR-tableau and a horizontal strip.*

Proof. Let Γ^e consist of the following columns of Γ : Start with the first occurrence of a column which contains the largest entry, say ℓ , in Γ . Then, for each $i = \ell - 1, \ell - 2, \dots, 1$, take the first column containing an i on the right hand side of the previously selected column. The remaining columns form Γ' . \square

COROLLARY 2.10. *Every LR-tableau which is a horizontal strip can be realized as the LR-tableau of a direct sum of poles and empty pickets.*

Proof. Use Lemma 2.9 repeatedly to decompose the given LR-tableau Γ into LR-tableaux of poles and empty pickets. According to Lemma 2.3, Γ is the LR-tableau of the direct sum of those poles and empty pickets. \square

LEMMA 2.11. *Suppose $\Gamma, \tilde{\Gamma}$ are LR-tableaux of the same shape, both horizontal strips, such that $\tilde{\Gamma}$ is obtained from Γ by a single box move which exchanges the entries u and v with $u < v$ in such a way that $\Gamma <_{\text{box}} \tilde{\Gamma}$. Then the columns of $\Gamma, \tilde{\Gamma}$ can be partitioned into LR-tableaux*

$$\Gamma = \Gamma' \cup \Gamma'' \cup \Gamma''', \quad \tilde{\Gamma} = \tilde{\Gamma}' \cup \tilde{\Gamma}'' \cup \Gamma'''$$

with the following properties:

1. $\Gamma', \Gamma'', \tilde{\Gamma}', \tilde{\Gamma}''$ are LR-tableaux of extended poles, they are horizontal strips and have no empty columns;
2. Γ''' is an LR-tableau which is a horizontal strip;
3. Γ' and $\tilde{\Gamma}'$ differ only in the length of one column containing an \boxed{u} , and neither tableau has a column of length in between;
4. Γ'' and $\tilde{\Gamma}''$ differ only in the length of one column containing a \boxed{v} , and neither tableau has a column of length in between; and
5. $\tilde{\Gamma}' \cup \tilde{\Gamma}''$ is obtained from $\Gamma' \cup \Gamma''$ by a single box move which exchanges the entries u and v .

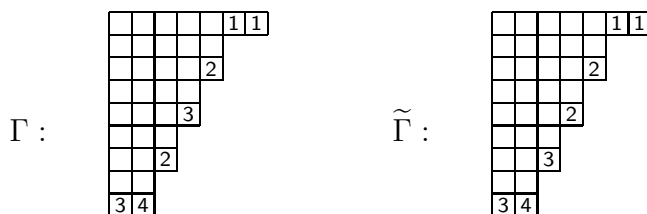
Proof. Suppose the entries involved in the box move are u and v with $u < v$. Denote by $c_u, c_v, \tilde{c}_u, \tilde{c}_v$ the columns in Γ and $\tilde{\Gamma}$, respectively, which contain those entries. Since $\Gamma <_{\text{box}} \tilde{\Gamma}$, column c_u occurs on the left of column c_v in Γ and column \tilde{c}_u occurs on the right of \tilde{c}_v in $\tilde{\Gamma}$. Let $\hat{\Gamma}$ be obtained from Γ by replacing column c_v by \tilde{c}_v . (So $\hat{\Gamma}$ has two long columns.) By arranging the columns in order, $\hat{\Gamma}$ becomes an LR-tableau, because Γ and $\tilde{\Gamma}$ are LR-tableaux. Using the algorithm in Lemma 2.9, repeatedly split off the LR-tableau Γ^e of an extended pole from $\hat{\Gamma}$. Three cases are possible.

- If Γ^e does contain neither column c_u nor column \tilde{c}_v , take Γ^e as part of Γ''' .

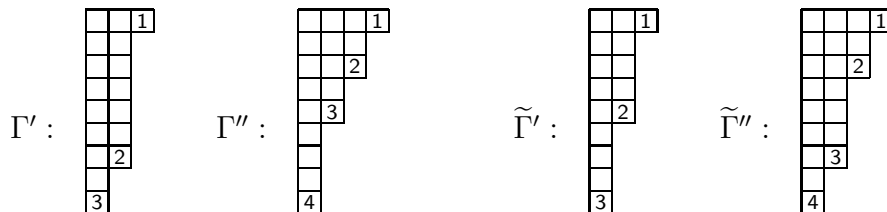
- If the algorithm encounters c_u , disregard in Γ^e all columns on the right hand side of c_u . Instead, take as next column the first occurrence of $u - 1$ in $\widehat{\Gamma}$ on the right hand side of the (missing) column c_v , and then continue the algorithm. This yields the LR-tableau Γ' which is a horizontal strip, has no multiple entries and no empty columns. By construction, there is no column of length between the lengths of c_u and c_v . Let $\widetilde{\Gamma}'$ be obtained from Γ' by replacing c_u by \widetilde{c}_u .
- Similarly, if the algorithm encounters \widetilde{c}_v , disregard in Γ^e all columns on the right hand side of \widetilde{c}_v . Instead, take as next column the first occurrence of $v - 1$ in $\widehat{\Gamma}$ on the right hand side of the (missing) column c_v , and continue. This extended pole is $\widetilde{\Gamma}''$. Let Γ'' be obtained from $\widetilde{\Gamma}''$ by replacing \widetilde{c}_v by c_v .

In each case, put $\hat{\Gamma} := \hat{\Gamma} \setminus \Gamma^e$. By construction, the new $\hat{\Gamma}$ is an LR-tableau which is a horizontal strip. Repeat, starting with the algorithm which computes the new Γ^e , until there are no entries left in $\hat{\Gamma}$.

Example: We will revisit the following example in Section 5. The tableau $\tilde{\Gamma}$ is obtained from Γ via a box move.



The partition into poles is given by the LR-tableaux below. Here, Γ''' is empty.



3. The boundary relation and its properties

In this section we present properties of the boundary relation defined in Formula 1.1.

3.1. The boundary relation is a preorder

We show that the boundary relation for LR-tableaux is a preorder which implies the dominance order.

For the proof of Part (a) in Theorem 1.2 we give a lemma.

LEMMA 3.1. *Suppose A, B are vector spaces and $\mathcal{M} \subseteq \text{Hom}_k(A, B)$ is a set of monomorphisms. For subspaces $U \subseteq A, V \subseteq B$ and a natural number n , the set*

$$\{f \in \mathcal{M}; \dim(f(U) \cap V) \geq n\}$$

is closed in \mathcal{M} .

Proof. Recall that for a natural number m , the condition $\text{rank}(f) > m$ defines an open subset in $\text{Hom}_k(A, B)$ since it is given by the non-vanishing of a minor in the matrix representing f . By restricting that matrix to a basis for U and a basis for the complement of V , we see that the condition $\dim \frac{f(U)+V}{V} > m$ also defines an open subset in $\text{Hom}_k(A, B)$. Let now $m = \dim U - n$. From the isomorphism $\frac{f(U)+V}{V} \cong \frac{f(U)}{f(U) \cap V}$ we obtain that the subset defined by $\dim \frac{f(U)}{f(U) \cap V} > m$ is open, in particular it is open when restricted to \mathcal{M} . Since on \mathcal{M} , all spaces $f(U)$ have the same dimension (f is a monomorphism), the condition is equivalent to

$$\dim f(U) \cap V < \dim f(U) - m = n.$$

The complementary condition $\dim f(U) \cap V \geq n$ defines a closed subset of \mathcal{M} . \square

PROPOSITION 3.2. *For all natural numbers i, ℓ, n , the subset*

$$\bigcup \{ \mathbb{V}_\Gamma : \Gamma \text{ satisfies } (\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell \geq n \}$$

in $\mathbb{V}_{\alpha, \gamma}^\beta(k)$ is closed.

Proof. Denote by P^ℓ the $k[T]$ -module $k[T]/T^\ell$ with only one Jordan block, so

$$B/f(T^i A) = \bigoplus_j P^{\gamma_j^{(i)}},$$

where $B = N_\beta$, $A = N_\alpha$ and $f \in \mathbb{V}_{\alpha, \gamma}^\beta$. Recall that $\dim \operatorname{Hom}_{k[T]}(P^\ell, P^m) = \min\{\ell, m\} = \dim \frac{P^\ell}{T^m P^\ell}$. Thus:

$$\begin{aligned} (\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell &= \sum_j \min\{\gamma_j^{(i)}, \ell\} \\ &= \dim \operatorname{Hom}_{k[T]}(B/f(T^i A), P^\ell) \\ &= \dim \frac{B/f(T^i A)}{T^\ell(B/f(T^i A))} \\ &= \dim \frac{B/f(T^i A)}{(T^\ell B + f(T^i A))/f(T^i A)} \end{aligned}$$

Using the isomorphism $\frac{T^\ell B + f(T^i A)}{f(T^i A)} \cong \frac{T^\ell B}{T^\ell B \cap f(T^i A)}$ we obtain

$$(\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell = \dim B - \dim f(T^i A) - \dim T^\ell B + \dim T^\ell B \cap f(T^i A).$$

Since $\dim B - \dim f(T^i A) - \dim T^\ell B = c$ is constant on $\mathbb{V}_{\alpha, \gamma}^\beta$, Lemma 3.1 implies that the set

$$\bigcup \{ \mathbb{V}_\Gamma : (\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell \geq n \} = \{ f \in \mathbb{V}_{\alpha, \gamma}^\beta : \dim T^\ell B \cap f(T^i A) \geq n - c \}$$

is a closed subset of $\mathbb{V}_{\alpha, \gamma}^\beta$. \square

We can now show that the boundary relation implies the dominance order.

Proof of Part (a) of Theorem 1.2. We assume that $\Gamma \not\prec_{\text{dom}} \tilde{\Gamma}$ and show that $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma = \emptyset$. By assumption, there exist i, ℓ such that

$$n = (\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell > (\tilde{\gamma}^{(i)})'_1 + \cdots + (\tilde{\gamma}^{(i)})'_\ell$$

holds. By the proposition, $\mathbb{U} = \bigcup \{ \mathbb{V}_{\tilde{\Gamma}} : (\hat{\gamma}^{(i)})'_1 + \cdots + (\hat{\gamma}^{(i)})'_\ell \geq n \}$ is a closed subset of $\mathbb{V}_{\alpha, \gamma}^\beta$ such that

$$\mathbb{V}_\Gamma \subseteq \mathbb{U} \quad \text{and} \quad \mathbb{U} \cap \mathbb{V}_{\tilde{\Gamma}} = \emptyset.$$

Thus, $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_\Gamma = \emptyset$. \square

As a consequence we obtain:

COROLLARY 3.3. *The boundary relation is a preorder (i.e. reflexive and antisymmetric). \square*

We conclude this section with a result for later use.

LEMMA 3.4. *Suppose $f, g : N_\alpha \rightarrow N_\beta$ are objects in $\mathbb{V}_{\alpha, \gamma}^\beta$. Let W be a subspace of N_β which is invariant under all automorphisms of N_β as a $k[T]$ -module. If $\mathcal{O}_f \subset \overline{\mathcal{O}}_g$ then*

$$\dim \operatorname{Im} f \cap W \geq \dim \operatorname{Im} g \cap W.$$

Examples of possible invariant submodules of N_β are the powers of the radical $T^r N_\beta$, powers of the socle $T^{-s} 0$, and their intersections $T^r N_\beta \cap T^{-s} 0$.

Proof. Let $h_\lambda : N_\alpha \rightarrow N_\beta$ be a one-parameter family of objects in $\mathbb{V}_{\alpha, \gamma}^\beta$ such that $h_\lambda \cong g$ for $\lambda \neq 0$ and $h_0 \cong f$. Put $n = \dim \operatorname{Im} g \cap W$.

Any isomorphism $h_\lambda \cong g$ ($\lambda \neq 0$) induces an isomorphism $\operatorname{Im} h_\lambda \cap W \cong \operatorname{Im} g \cap W$ since W is invariant under automorphisms of N_β . By Lemma 3.1, the set

$$\{h \in \mathbb{V}_{\alpha, \gamma}^\beta : \dim \operatorname{Im} h \cap W \geq n\}$$

is closed in $\mathbb{V}_{\alpha, \gamma}^\beta$, so with h_λ , $\lambda \neq 0$, also h_0 is in the set. This shows $\dim \operatorname{Im} f \cap W = \dim \operatorname{Im} h_0 \cap W \geq n$. \square

3.2. The boundary relation and dominance

We have seen in Section 3.1 that the boundary relation implies the dominance relation. Here we give an example that in general, the boundary relation is strictly stronger than the dominance relation.

In this and in the following section, we determine all isomorphism types of objects which realize a given tableau that has at most 4 rows. Such objects occur in the category $\mathcal{S}(4)$ studied in [21, (6.4)] of all pairs consisting of a nilpotent linear operator with nilpotency index at most 4 and an invariant subspace.

LEMMA 3.5. *Each object in the category $\mathcal{S}(4)$ is a direct sum of indecomposables. There are 20 indecomposable objects, up to isomorphism: Four empty pickets P_0^1, \dots, P_0^4 , fifteen poles $P(S)$, where S is a non-empty subset of*

$\{0, 1, 2, 3, 4\}$, and a remaining object X which has the property that the invariant subspace has two Jordan blocks:

$$X : \begin{array}{|c|c|} \hline & \\ \hline \bullet & \bullet \\ \hline & \bullet \\ \hline & \\ \hline \end{array} \quad \Gamma_X : \begin{array}{|c|c|} \hline & 1 \\ \hline & 2 \\ \hline 1 & \\ \hline 3 & \\ \hline \end{array}$$

□

For the computation of tableaux recall that the tableau of a pole $P(x_1, \dots, x_k)$ has entries $1, \dots, k$ in rows x_1+1, \dots, x_k+1 , see Lemma 2.7; and that the LR-tableau of a direct sum is obtained by merging the rows of the LR-tableaux of the summands, see Lemma 2.3.

Example: For $\alpha = (3, 1)$, $\beta = (4, 3, 1)$, $\gamma = (3, 1)$, there are two LR-tableaux of shape (α, β, γ) :

$$\Gamma_1 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \\ \hline & 2 & \\ \hline 3 & & \\ \hline \end{array} \quad \Gamma_2 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 & \\ \hline & 3 & \\ \hline 1 & & \\ \hline \end{array}$$

We determine the possible isomorphism types of embeddings which have LR-tableaux Γ_1 and Γ_2 , respectively. For each tableau, there is only one realization, up to isomorphy.

$$M_1 = P_3^4 \oplus P_0^3 \oplus P_1^1 : \begin{array}{|c|} \hline \\ \hline \bullet \\ \hline \\ \hline \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \quad M_2 = P_1^4 \oplus P_3^3 \oplus P_0^1 : \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \bullet \\ \hline \\ \hline \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

There are no other realizations: Any such embedding occurs in the category $\mathcal{S}(4)$, so Lemma 3.5 can be used. Considering the LR-tableau for X , this module cannot occur as a summand. Hence any realization is a direct sum of poles and empty pickets. Note that the pole $P(0, 2, 3)$ cannot occur in a decomposition for Γ_1 since this would require that P_1^2 is a summand, which is not possible since there is no column of length 2 in Γ_1 . Since poles are determined by the rows in which their entries occur, there are no other choices.

As a consequence, the varieties \mathbb{V}_{Γ_1} and \mathbb{V}_{Γ_2} have the same dimension, and each consists of only one orbit. Hence

$$\mathbb{V}_{\Gamma_1} \cap \overline{\mathbb{V}}_{\Gamma_2} = \emptyset = \mathbb{V}_{\Gamma_2} \cap \overline{\mathbb{V}}_{\Gamma_1}.$$

Thus, Γ_1 and Γ_2 are not in boundary relation, but clearly $\Gamma_1 >_{\text{dom}} \Gamma_2$.

3.3. The boundary relation may not be transitive

In general, the boundary relation given by

$$\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma} \neq \emptyset$$

is not transitive. In this section, we provide an example.

Example: Let $\alpha = (3, 1)$, $\beta = (4, 3, 2, 1)$, $\gamma = (3, 2, 1)$. There are three LR-tableaux:

$$\Gamma_1 : \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 1 & \\ \hline & 2 & & \\ \hline 3 & & & \\ \hline \end{array} \quad \Gamma_2 : \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & \\ \hline & 1 & & \\ \hline 3 & & & \\ \hline \end{array} \quad \Gamma_3 : \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & \\ \hline & 3 & & \\ \hline 1 & & & \\ \hline \end{array}$$

Distributed over those three tableaux are five pairwise nonisomorphic embeddings which can be determined using Lemma 3.5.

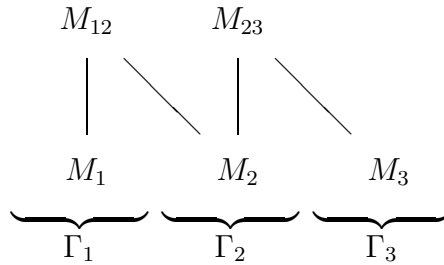
$$\begin{aligned} M_1 &= P_3^4 \oplus P_0^3 \oplus P_0^2 \oplus P_1^1, \\ M_{12} &= P(0, 2, 3) \oplus P_0^3 \oplus P_1^2, \\ M_2 &= X \oplus P_0^3 \oplus P_0^1, \\ M_{23} &= P(0, 1, 3) \oplus P_1^3 \oplus P_0^1, \\ M_3 &= P_1^4 \oplus P_3^3 \oplus P_0^2 \oplus P_0^1 \end{aligned}$$

The notation is such that M_i or M_{ix} has LR-tableau Γ_i .

For the convenience of the reader, we picture the poles $P(0, 2, 3)$ and $P(0, 1, 3)$ and their LR-tableaux.

$$P(0, 2, 3) : \begin{array}{|c|c|} \hline & \\ \hline \bullet & \bullet \\ \hline & \\ \hline & \\ \hline \end{array} \quad \Gamma_P : \begin{array}{|c|c|} \hline & 1 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array} \quad P(0, 1, 3) : \begin{array}{|c|c|} \hline & \\ \hline \bullet & \bullet \\ \hline & \\ \hline & \\ \hline \end{array} \quad \Gamma_P : \begin{array}{|c|c|} \hline & 1 \\ \hline & 2 \\ \hline 3 & \\ \hline \end{array}$$

We show that the containment relation of orbit closures is as follows.



The short exact sequence

$$0 \longrightarrow P_1^2 \longrightarrow M_2 \longrightarrow P(0, 2, 3) \oplus P_0^3 \longrightarrow 0$$

shows that $\mathcal{O}(M_{12}) \subset \overline{\mathcal{O}}(M_2)$ (since the ext-order implies the degeneration order, see Section 4). Hence $\mathbb{V}_{\Gamma_1} \cap \overline{\mathbb{V}}_{\Gamma_2} \neq \emptyset$ and $\Gamma_1 >_{\text{bound}}^* \Gamma_2$.

Similarly, the short exact sequence

$$0 \longrightarrow P_1^3 \longrightarrow M_3 \longrightarrow P(0, 1, 3) \oplus P_0^1 \longrightarrow 0$$

shows that $\mathcal{O}(M_{23}) \subset \overline{\mathcal{O}}(M_3)$, hence $\mathbb{V}_{\Gamma_2} \cap \overline{\mathbb{V}}_{\Gamma_3} \neq \emptyset$ and $\Gamma_2 >_{\text{bound}}^* \Gamma_3$.

However, $\mathbb{V}_{\Gamma_1} \cap \overline{\mathbb{V}}_{\Gamma_3} = \emptyset$. The only possible orbit in the intersection is $\mathcal{O}(M_{12})$, since there are only two orbits in \mathbb{V}_{Γ_1} , and since the other orbit $\mathcal{O}(M_1)$ has the same dimension as $\mathbb{V}_{\Gamma_3} = \mathcal{O}(M_3)$.

Note that the module $M_{12} = (U \subset V)$ has the property that $\dim U \cap T^2V \cap \text{soc } V = 1$, while for the module M_3 , the corresponding dimension is 2. It follows from Lemma 3.4 with $W = T^2V \cap \text{soc } V$ that $\mathcal{O}(M_{12}) \not\subseteq \overline{\mathcal{O}}(M_3)$.

This finishes the example which illustrates that in general, the condition for LR-tableaux that $\mathbb{V}_{\tilde{\Gamma}} \cap \overline{\mathbb{V}}_{\Gamma} \neq \emptyset$ may not define a partial order. \square

4. The algebraic orders for LR-tableaux

For modules of a fixed dimension over a finite dimensional algebra the three partial orders

$$\leq_{\text{ext}}, \quad \leq_{\text{deg}}, \quad \leq_{\text{hom}}$$

have been studied extensively, see for example [1, 2, 20, 9, 23]. In particular, the partial orders are available for invariant subspaces in $\mathbb{V}_{\alpha, \gamma}^\beta$, see [11, Section 3.2]. For the convenience of the reader we recall these definitions. Let $f, g \in \mathbb{V}_{\alpha, \gamma}^\beta$.

- The relation $f \leq_{\text{ext}} g$ holds if there exist embeddings h_i, u_i, v_i in $\mathbb{V}_{\alpha, \gamma}^\beta$ and short exact sequences $0 \rightarrow u_i \rightarrow h_i \rightarrow v_i \rightarrow 0$ of embeddings such that $f \cong h_1$, $u_i \oplus v_i \cong h_{i+1}$ for $1 \leq i \leq s$, and $g \cong h_{s+1}$, for some natural number s .
- The relation $f \leq_{\text{deg}} g$ holds if $\mathcal{O}_g \subseteq \overline{\mathcal{O}}_f$ in $\mathbb{V}_{\alpha, \gamma}^\beta(k)$.

- The relation $f \leq_{\text{hom}} g$ holds if

$$[f, h] \leq [g, h]$$

for any embedding h in $\mathbb{V}_{\alpha, \gamma}^\beta$, where $[f, h]$ denotes the dimension of the linear space $\text{Hom}(f, h)$ of all homomorphisms of embeddings.

They induce three preorders on the set $\mathcal{T}_{\alpha, \gamma}^\beta: \leq_{\text{ext}}^*, \leq_{\text{deg}}^*, \leq_{\text{hom}}^*$ which, as we will see, are in fact partial orders. Namely, each of the relations listed implies the dominance order.

Definition: Suppose $\Gamma, \tilde{\Gamma}$ are two LR-tableaux of shape (α, β, γ) . We write $\Gamma \leq_{\text{ext}}^* \tilde{\Gamma}$ ($\Gamma \leq_{\text{deg}}^* \tilde{\Gamma}$; $\Gamma \leq_{\text{hom}}^* \tilde{\Gamma}$) if there is a sequence

$$\Gamma = \Gamma^{(0)}, \Gamma^{(1)}, \dots, \Gamma^{(s)} = \tilde{\Gamma}$$

such that for each $1 \leq i \leq s$ there are $f \in \mathbb{V}_{\Gamma^{(i-1)}}$, $g \in \mathbb{V}_{\Gamma^{(i)}}$ with $f \leq_{\text{ext}} g$ ($f \leq_{\text{deg}} g$; $f \leq_{\text{hom}} g$).

It follows from the corresponding properties for modules that:

- $\Gamma \leq_{\text{ext}}^* \tilde{\Gamma}$ implies $\Gamma \leq_{\text{deg}}^* \tilde{\Gamma}$ and
- $\Gamma \leq_{\text{deg}}^* \tilde{\Gamma}$ implies $\Gamma \leq_{\text{hom}}^* \tilde{\Gamma}$.

Also, it is clear from the definitions that

- $\Gamma \leq_{\text{deg}}^* \tilde{\Gamma}$ implies $\Gamma \leq_{\text{bound}}^* \tilde{\Gamma}$.

We have seen in Section 3.1 that the boundary relation implies the dominance order \leq_{dom} . In the following section we show that also the hom-relation implies the dominance order.

4.1. Hom-relation implies dominance order

We start with an abstract result.

Denote by \mathcal{N} the category $\text{mod } k[T]_{(T)}$ of all nilpotent linear operators, and by $\mathcal{S} = \mathcal{S}(k[T]_{(T)})$ the category of all invariant subspaces. For each $i \in \mathbb{N}$, there is a pair of functors

$$\begin{aligned} R_i : \mathcal{S} &\rightarrow \mathcal{N}, \quad (A \subset B) \mapsto \frac{B}{T^i A} \\ L_i : \mathcal{N} &\rightarrow \mathcal{S}, \quad X \mapsto (\text{soc}^i X \subset X). \end{aligned}$$

LEMMA 4.1. *For each $i \in \mathbb{N}$, the functors R_i, L_i form an adjoint pair.*

Proof. Given an operator $X \in \mathcal{N}$ and an invariant subspace $(A \subset B) \in \mathcal{S}$, we need to show that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{S}}((A \subset B), L_i(X)) \cong \mathrm{Hom}_{\mathcal{N}}(R_i(A \subset B), X).$$

A morphism in \mathcal{S} is given by a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & \mathrm{soc}^i X \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & X \end{array}$$

It gives rise to the commutative diagram:

$$\begin{array}{ccc} \mathrm{rad}^i A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & X \end{array}$$

Hence we obtain a morphism in \mathcal{N} :

$$\bar{f} : \frac{B}{\mathrm{rad}^i A} \longrightarrow X.$$

Conversely, the morphism in \mathcal{N} gives rise to a commutative diagram and hence to a morphism in \mathcal{S} . Clearly, the two constructions are inverse to each other. \square

We recognize that the objects of the form $P_i^\ell = L_i(P^\ell)$ are pickets.

PROPOSITION 4.2. *Suppose the objects $(A \subset B)$ and $(\tilde{A} \subset \tilde{B})$ have LR-tableaux Γ and $\tilde{\Gamma}$, respectively. The following assertions are equivalent:*

1. $\Gamma \leq_{\mathrm{dom}} \tilde{\Gamma}$
2. *For each picket P_i^ℓ the inequality holds:*

$$\dim \mathrm{Hom}_{\mathcal{S}}((A \subset B), P_i^\ell) \leq \dim \mathrm{Hom}_{\mathcal{S}}((\tilde{A} \subset \tilde{B}), P_i^\ell)$$

Proof. By the definition given in Section 2.1, the condition $\Gamma \leq_{\text{dom}} \tilde{\Gamma}$ is equivalent to

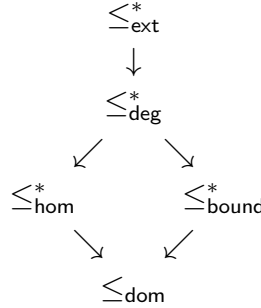
$$(\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell \leq (\tilde{\gamma}^{(i)})'_1 + \cdots + (\tilde{\gamma}^{(i)})'_\ell \quad \text{for each } i \text{ and } \ell.$$

Let i and ℓ be natural numbers. We obtain from Lemma 4.1 and from the equality in the proof of Proposition 3.2 that

$$(\gamma^{(i)})'_1 + \cdots + (\gamma^{(i)})'_\ell = \dim \text{Hom}_{\mathcal{N}}(B/T^i A, P^\ell) = \dim \text{Hom}_{\mathcal{S}}((A \subset B), P_i^\ell)$$

The claim follows from this and from the corresponding equality for $(\tilde{A} \subset \tilde{B})$. \square

It follows that the hom-relation implies the dominance order. Here is what we have. So far, we have not imposed any conditions on the triple (α, β, γ) .



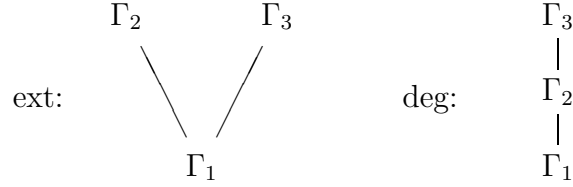
4.2. The ext- and deg-relations are not equivalent

It is well-known that for modules, the ext-relation \leq_{ext} implies the deg-relation \leq_{deg} . In general for modules, the converse is not the case. Here we give an example for embeddings of linear operators.

Example: For $\alpha = (4, 2)$, $\beta = (6, 4, 2)$, $\gamma = (4, 2)$, there are three LR-tableaux:

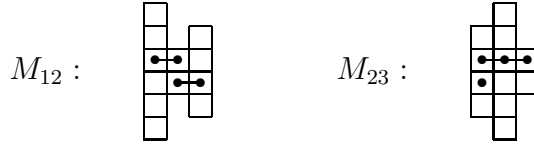
$$\Gamma_1 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & 2 \\ \hline & 1 & \\ \hline & 2 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array} \quad \Gamma_2 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & 2 \\ \hline & 1 & \\ \hline & 3 & \\ \hline 2 & & \\ \hline 4 & & \\ \hline \end{array} \quad \Gamma_3 : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & 2 \\ \hline & 3 & \\ \hline & 4 & \\ \hline 1 & & \\ \hline 2 & & \\ \hline \end{array}$$

We show that the partial orders given by \leq_{ext} and \leq_{deg} are as follows:

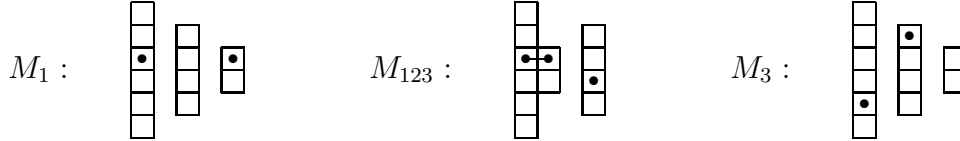


(In each case, Γ_1 is the largest element in the poset.)

First we describe the embeddings which realize the tableaux. From [21] we know that there is a one-parameter family of indecomposable embeddings $M_2(\lambda)$ occurring on the mouths of the homogeneous tubes with tubular index 0; they all have type Γ_2 . There are two additional indecomposables, they occur in the tube of circumference 2 at index 0; the modules are dual to each other and have type Γ_1 and Γ_2 , respectively. We sketch the modules, using the conventions as in [21].



In addition, there are three decomposable configurations; note that M_1 is the dual of M_3 while M_{123} is self dual.



The modules $M_1 = P_4^6 \oplus P_0^4 \oplus P_2^2$ and $M_{123} = P(0, 1, 4, 5) \oplus P_2^4$ have type Γ_1 , and $M_3 = P_2^6 \oplus P_4^4 \oplus P_0^2$ has type Γ_3 .

We claim that there are no further isomorphism types of objects in $\mathbb{V}_{\alpha, \gamma}^\beta$.

For finite fields, the Hall polynomial $g_{\alpha, \gamma}^\beta$ counts the number of submodules of N_β which are isomorphic to N_α and have factor N_γ . For each of the isomorphism types of embeddings (that is, M_1 , M_{12} , M_{123} , $M_2(\lambda)$ ($\lambda \neq 0, 1$), M_{23} , M_3), we can count the corresponding numbers of submodules of N_β . It is straightforward to verify that the sum, taken over the isomorphism types, is exactly $g_{\alpha, \gamma}^\beta$.

For algebraically closed fields, the embeddings M_1 , M_{123} , M_3 are sums of exceptional objects in the covering category $\mathcal{S}(\widetilde{6})$ studied in [21], the others

are indecomposable non-exceptional objects. The $M_2(\lambda)$ occur in the homogeneous tubes, M_{12} and M_{23} in the tube of circumference 2 in the tubular family of index 0; the remaining tubes of index 0 are pictured in [21, (2.3)], they contain no non-exceptional objects in $\mathbb{V}_{\alpha,\gamma}^\beta$. All non-exceptional objects in tubes of index different from 0 have higher dimension. It follows that each remaining object in $\mathbb{V}_{\alpha,\gamma}^\beta$ is a direct sum of exceptional modules. Each exceptional object X is determined uniquely by its dimension vector in $\mathcal{S}(\tilde{6})$ and can be realized over any field. The dimension of the homomorphism spaces $\text{Hom}(P, X)$ where P is a picket, and hence the LR-tableau for X ([22]) do not depend on the base field. Hence M_1 , M_{123} and M_3 are the only objects in $\mathbb{V}_{\alpha,\gamma}^\beta$ which have an exceptional direct summand.

We determine the ext-order and the deg-order on $\mathcal{T}_{\alpha,\gamma}^\beta$. Consider the short exact sequences

$$0 \longrightarrow P_2^4 \longrightarrow M_{23} \longrightarrow P(0, 1, 4, 5) \longrightarrow 0$$

and

$$0 \longrightarrow P_2^4 \longrightarrow M_3 \longrightarrow P(0, 1, 4, 5) \longrightarrow 0.$$

In each, the sum of the end terms is M_{123} . It follows that $\Gamma_1 \geq_{\text{ext}} \Gamma_2$ and $\Gamma_1 \geq_{\text{ext}} \Gamma_3$, respectively. Note that $\Gamma_2 \not\geq_{\text{ext}} \Gamma_3$ since there is no decomposable module of type Γ_2 .

Since the ext-relation implies the deg-relation, it remains to show that $\Gamma_2 \geq_{\text{deg}} \Gamma_3$. As mentioned, the modules M_1 and M_3 are dual to each other, so their orbits have the same dimension. As $\mathcal{O}_{M_3} = \mathbb{V}_{\Gamma_3}$, and since all varieties given by LR-tableaux are irreducible of the same dimension, it follows that \mathcal{O}_{M_1} is dense in \mathbb{V}_{Γ_1} . In particular, \mathcal{O}_{M_1} contains $\mathcal{O}_{M_{12}}$ in its closure. Applying duality again, we obtain that \mathcal{O}_{M_3} contains $\mathcal{O}_{M_{23}}$ in its closure. Thus, $\mathcal{O}_{M_{23}}$ is in the closure of \mathbb{V}_{Γ_3} . \square

5. Box-relation and ext-relation

In this section we assume that α, β, γ are partitions such that $\beta \setminus \gamma$ is a horizontal strip. We show that the box-order implies the ext-relation (Proposition 1.4). As a consequence, we can complete the proof of Theorem 1.5.

The statement $\Gamma <_{\text{ext}} \tilde{\Gamma}$ is a consequence of the following two facts which we will show in a more general set-up in Section 5.3:

1. There is a short exact sequence $0 \rightarrow \tilde{X} \rightarrow Y \rightarrow \tilde{Z} \rightarrow 0$.
2. The LR-tableau for Y is Γ .

Note that in the above example, the modules $X, Z, \tilde{X}, \tilde{Z}$ are all indecomposable. We present a second example in which X, \tilde{X} and \tilde{Z} are poles, hence indecomposable, but Z is the direct sum of a pole and an empty picket.

Example:

$$\Gamma : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 & \\ \hline & & \\ \hline & & \\ \hline 1 & & \\ \hline \end{array} \quad \tilde{\Gamma} : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \\ \hline & & \\ \hline & & \\ \hline & & 2 \\ \hline \end{array}$$

Clearly, $\Gamma <_{\text{box}} \tilde{\Gamma}$. Here, Lemma 2.11 yields the following decomposition.

$$\Gamma' : \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline 1 \\ \hline \end{array} \quad \Gamma'' : \begin{array}{|c|c|} \hline & 1 \\ \hline 2 & \\ \hline \end{array} \quad \tilde{\Gamma}' : \begin{array}{|c|} \hline \\ \hline 1 \\ \hline \end{array} \quad \tilde{\Gamma}'' : \begin{array}{|c|c|} \hline & 1 \\ \hline & \\ \hline & \\ \hline & \\ \hline 2 & \\ \hline \end{array}$$

The corresponding poles are:

$$X = P(4) : \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \bullet \\ \hline \end{array} \quad Z = P(0, 1) \oplus P_0^1 : \begin{array}{|c|} \hline \bullet \\ \hline \\ \hline \end{array} \square \quad \tilde{X} = P(1) : \begin{array}{|c|} \hline \\ \hline \bullet \\ \hline \end{array} \quad \tilde{Z} = P(0, 4) : \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \bullet & \bullet \\ \hline \end{array}$$

In this example we can put $Y = X \oplus Z$. One can check that Y occurs as the middle term of a short exact sequence $0 \rightarrow \tilde{X} \rightarrow Y \rightarrow \tilde{Z} \rightarrow 0$.

$$Y : \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & \bullet & \\ \hline \bullet & & \square \\ \hline \end{array}$$

5.2. Gradings

To present a general construction for the exact sequence which witnesses the implication from \leq_{box} to \leq_{ext} , we will work in the category of graded embeddings.

Let $\text{mod}_0^{\mathbb{Z}}k[T]$ denote the category of graded nilpotent $k[T]$ -modules, where T has degree 1. Each indecomposable object $P^m[d]$ has support given by an interval $[d, d + m - 1]$; the object is generated by a homogeneous element in degree d . Clearly, by forgetting the grading, $P^m[d]$ is just the $k[T]$ -module $k[T]/(T^m)$.

For $B \in \text{mod}_0^{\mathbb{Z}}k[T]$, we denote the vector space in degree ℓ by B_ℓ . By $\mathcal{S}^{\mathbb{Z}}$ we denote the category of graded embeddings between objects in $\text{mod}_0^{\mathbb{Z}}k[T]$. The graded shift $[1]$ and its powers are self-equivalences on $\text{mod}_0^{\mathbb{Z}}k[T]$ and on $\mathcal{S}^{\mathbb{Z}}$. The d -fold shift of an object $B \in \text{mod}_0^{\mathbb{Z}}k[T]$ is given by $B[d]_\ell = B_{\ell-d}$.

Suppose Γ is an LR-tableau that occurs in a pole decomposition as in Lemma 2.11, so Γ is a horizontal strip with no multiple entries and no empty columns. Assume that the type of Γ is (α, β, γ) where $\beta = (\beta_1, \dots, \beta_t)$. We specify an extended pole $P(\Gamma)$ which is given by the following graded embedding ($A \subset B$). The grading is such that the submodule generator has degree 0. Thus, the i -th column of Γ (which contains the entry $t - i + 1$) will give rise to the summand $P^{\beta_i}[t - i + 1 - \beta_i]$ of B which has support $[t - i + 1 - \beta_i, t - i]$.

$$\begin{aligned} B &= \bigoplus_{i=1}^t P^{\beta_i}[(t - i + 1) - \beta_i] \\ a &= (T^{\beta_i - (t - i + 1)})_{i=1, \dots, t} = \sum_{i=1}^t g^{\beta_i} T^{\beta_i - (t - i + 1)} \\ A &= a k[T] \end{aligned}$$

Here, the elements g^{β_i} are the generators of the summands of B . The notation is defined since β has no multiple parts.

We obtain as the proof of Lemma 2.8:

LEMMA 5.1. *The extended pole $P(\Gamma)$ has LR-tableau Γ .* □

Here is an example of a graded pole.

Example: Consider the following LR-tableau Γ corresponding to the pole

$P(0, 2, 5)$.

$$\Gamma : \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & \\ \hline & 2 & \\ \hline & & \\ \hline & & \\ \hline & 3 & \\ \hline \end{array}$$

Note that $A = P^3$, $B = P^1 \oplus P^3[-1] \oplus P^6[-3]$, $a = (1, T, T^3)$. The grading for the pole $P(0, 2, 5)$ is as follows.

$$(A \subset B) : \begin{array}{|c|c|c|c|} \hline & & & \\ \hline -3 & & & \\ \hline -2 & & & \\ \hline -1 & & & \\ \hline 0 & \bullet & \bullet & \bullet \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline \end{array} \quad \left(\frac{A+T^4B}{T^4B} \subset \frac{B}{T^4B} \right) : \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array}$$

Remark: Note that the above pole is a horizontal strip. Also poles which are not horizontal strips can be graded, but the above formula has to be modified. To keep the notation simple, we work with extended poles which are horizontal strips. In this case in general, the grading is not unique, up to the shift, since the indecomposable summands can be moved against each other.

5.3. Exact sequences for horizontal strips

In order to prove Theorem 1.4, we assume the set-up from Lemma 2.11. Let α, β, γ be partitions such that $\beta \setminus \gamma$ is a horizontal strip. Let $\Gamma, \tilde{\Gamma}$ be LR-tableaux of type (α, β, γ) .

Assume that $\tilde{\Gamma}$ is obtained from Γ by a single box move so that $\Gamma <_{\text{box}} \tilde{\Gamma}$. Hence the LR-tableau Γ has two boxes \boxed{u} and \boxed{v} in rows r and s where $u < v$ and $r > s$, and the tableau $\tilde{\Gamma}$ is obtained from Γ by replacing the boxes \boxed{u} and \boxed{v} .

Our goal is to construct a short exact sequence

$$0 \longrightarrow \tilde{X} \longrightarrow Y \longrightarrow \tilde{Z} \longrightarrow 0$$

such that the LR-type of Y is Γ and the LR-type of $\tilde{X} \oplus \tilde{Z}$ is $\tilde{\Gamma}$. This shows $\Gamma \leq_{\text{ext}} \tilde{\Gamma}$.

By Lemma 2.11 we can partition Γ and $\tilde{\Gamma}$ as

$$\Gamma = \Gamma' \cup \Gamma'' \cup \Gamma''', \quad \tilde{\Gamma} = \tilde{\Gamma}' \cup \tilde{\Gamma}'' \cup \tilde{\Gamma}'''$$

such that Properties (1) through (5) are satisfied.

According to Property (2), the common part Γ''' is an LR-tableau, so Γ''' can be realized as the tableau of an embedding U ; in fact, since Γ''' is a horizontal strip, U can be taken as a direct sum of poles and empty pickets (Corollary 2.10). For the purpose of constructing the short exact sequence, we may assume that Γ''' is empty since the embedding U can be added later to both \tilde{X} and Y , say.

According to Property (1), the LR-tableaux Γ' , Γ'' , $\tilde{\Gamma}'$ and $\tilde{\Gamma}''$ are horizontal strips with no multiple entries and no empty columns. Hence they can be realized as LR-tableaux of graded extended poles (Section 5.2), which we denote as follows:

$$X = P(\Gamma')[v - u], \quad Z = P(\Gamma''), \quad \tilde{X} = P(\tilde{\Gamma}')[v - u], \quad \tilde{Z} = P(\tilde{\Gamma}'').$$

To refer to the subspace and to the ambient space of such an embedding, we write $X : (X_{\text{sub}} \subset X_{\text{amb}})$ etc.

Using Property (3), we see that the embedding \tilde{X} embeds into X as follows. Namely, Γ' and $\tilde{\Gamma}'$ differ only in one column which gives \tilde{X} a column that is by $r - s$ units shorter than the corresponding column in X . On the ambient spaces, the inclusion map is as follows.

$$\iota_{\tilde{X}, X} : \tilde{X}_{\text{amb}} \rightarrow X_{\text{amb}}, \quad g_{\tilde{X}}^\ell \mapsto \begin{cases} g_X^r T^{r-s} & \text{if } \ell = s \\ g_X^\ell & \text{otherwise} \end{cases}$$

Note that $\text{cok}(\iota_{\tilde{X}, X})$ is an empty picket of height $r - s$. Recalling the definition of the subspace generators in Section 5.2, $\iota_{\tilde{X}, X}$ maps $a_{\tilde{X}}$ to a_X , and hence is a morphism in $\mathcal{S}^{\mathbb{Z}}$.

Dually, we can use Property (4) to embed Z into \tilde{Z} with cokernel also an empty picket of height $r - s$.

$$\iota_{Z, \tilde{Z}} : Z_{\text{amb}} \rightarrow \tilde{Z}_{\text{amb}}, \quad g_Z^\ell \mapsto \begin{cases} g_{\tilde{Z}}^r T^{r-s} & \text{if } \ell = s \\ g_{\tilde{Z}}^\ell & \text{otherwise} \end{cases}$$

Similarly, $\iota_{Z, \tilde{Z}}$ maps a_Z to $a_{\tilde{Z}}$ and hence is a morphism in $\mathcal{S}^{\mathbb{Z}}$.

We can now introduce the module Y . Define the ambient space as the sum $Y_{\text{amb}} = X_{\text{amb}} \oplus Z_{\text{amb}}$ of graded modules, and let $Y_{\text{sub}} = (a_X, g_Z^s T^{s-u}) + (0, a_Z)$

so $(a_X, g_Z^s T^{s-u})$ is in degree $v - u$ and $(0, a_Z)$ is in degree 0. The maps in the short exact sequence

$$\mathcal{E} : \quad 0 \longrightarrow \tilde{X} \xrightarrow{\iota_{\tilde{X}, Y}} Y \xrightarrow{\pi_{Y, \tilde{Z}}} \tilde{Z} \longrightarrow 0$$

are the following.

$$\iota_{\tilde{X}, Y} : \quad \tilde{X}_{\text{amb}} \rightarrow Y_{\text{amb}}, \quad g_{\tilde{X}}^\ell \mapsto \begin{cases} g_X^r T^{r-s} + g_Z^s & \text{if } \ell = s \\ g_X^\ell & \text{otherwise} \end{cases}$$

This map has first component $\iota_{\tilde{X}, X}$ and is a homomorphism since $\iota_{\tilde{X}, Y}(a_{\tilde{X}}) = (a_X, g_Z^s T^{s-u})$.

$$\pi_{Y, \tilde{Z}} : Y_{\text{amb}} \rightarrow \tilde{Z}_{\text{amb}}, \quad g_X^\ell \mapsto \begin{cases} 0 & \text{if } \ell \neq r \\ -g_Z^r & \text{if } \ell = r \end{cases} \quad \text{and} \quad g_Z^\ell \mapsto \begin{cases} g_{\tilde{Z}}^\ell & \text{if } \ell \neq s \\ g_Z^r T^{r-s} & \text{if } \ell = s \end{cases}$$

This map has second component $\iota_{Z, \tilde{Z}}$ and is a homomorphism. Namely, $\pi_{Y, \tilde{Z}}(a_X, g_Z^s T^{s-u}) = 0$ and $\pi_{Y, \tilde{Z}}(0, a_Z) = \iota_{Z, \tilde{Z}}(a_Z) = a_{\tilde{Z}}$.

It is straightforward to verify that the sequence \mathcal{E} is exact. Namely, $\iota_{\tilde{X}, Y}$ is a monomorphism, $\pi_{Y, \tilde{Z}}$ is an epimorphism, and the composition is zero.

It remains to show that the LR-tableau for Y is Γ . We have already seen that $X \oplus Z$ has LR-tableau Γ . So we show that Y and $X \oplus Z$ have the same LR-tableau. We write $Y = (A \subset B)$ and $X \oplus Z = (C \subset D)$. Recall that by construction, $B = D$. We have seen above in Section 5.2 that we need to verify that for each degree d , and for all exponents ℓ, q the subspaces

$$(5.2) \quad \left(\frac{T^\ell A + T^q B}{T^q B} \right)_d \quad \text{and} \quad \left(\frac{T^\ell C + T^q D}{T^q D} \right)_d$$

have the same dimension. Note that the subspace generators are very similar: $A = (a_X, g_Z^s T^{s-u}) + (0, a_Z)$ versus $C = (a_X, 0) + (0, a_Z)$. Note that the subspaces differ only in degrees $v - u \leq d < v$. Moreover, the addition of $(0, g_Z^s T^{s-u})_d$ to the first generator $(a_X, 0)_d$ changes the dimension only if $(a_X, 0)_d \in (T^q B)_d$ and if $(0, g_Z^s T^{s-u})_d \notin (T^q B)_d$, which is only possible if $d = v - 1$. In this case $d = v - 1$, if $(a_X, 0) \in (T^q B)_d$ then $q < r$, and if $(0, g_Z^s T^{s-u}) \notin (T^q B)_d$ then $q \geq s$. By construction of B , the space $(B/T^q B)_d$ has dimension 1 for $s \leq q < r$. Note that whenever $(0, g_Z^s T^{s-u})_d$ is nonzero,

then so is $(0, a_Z)_d$. Hence also for $d = v - 1$, and any q, ℓ , the dimensions in (5.2) are equal.

This finishes the proof that Y has LR-tableau Γ , as desired, and hence the proof of Proposition 1.4.

Proof of Theorem 1.5. In view of the implications pictured above Theorem 1.3, it suffices to add that for LR-tableaux which are horizontal strips of the same shape, the box-order implies the ext-relation (Proposition 1.4) and that for horizontal and vertical strips, the domination order and the box-order are equivalent ([13], see Theorem 2.2). \square

REFERENCES

- [1] K. Bongartz, *On degenerations and extensions of finite dimensional modules*, Adv. Math. **121** (1996), 245–287.
- [2] K. Bongartz, *Degenerations for representations of tame quivers*, Ann. Sci. Ec. Norm. Super. **28** (1995), 647–668.
- [3] B. Deng and J. Du, *Monomial bases for quantum affine \mathfrak{sl}_n* , Adv. Math. **191** (2005), 276–304.
- [4] B. Deng, J. Du and A. Mah, *Presenting degenerate Ringel-Hall algebras of cyclic quivers*, J. Pure Appl. Algebra **214** (2010), 1787–1799.
- [5] M. Gerstenhaber, *On dominance and varieties of commuting matrices*, Ann. of Math. (2) **73** (1961), 324–348.
- [6] W. Hesselink, *singularities in the nilpotent scheme of a classical group*, Trans. Amer. Math. Soc. **222** (1976), 1–32.
- [7] I. Kaplansky, *Infinite abelian groups*, The University of Michigan Press, Ann Arbor (1954), 5th printing (1965).
- [8] T. Klein, *The multiplication of Schur-functions and extensions of p -modules*, J. Lond. Math. Soc. **43** (1968), 280–284.

- [9] J. Kosakowska, *Degenerations in a class of matrix problems and projective modules*, J. Algebra **263** (2003), 262–277.
- [10] J. Kosakowska, *Generic extensions of nilpotent $k[T]$ -modules, monoids of partitions and constant terms of Hall polynomials*, Coll. Math. **128** (2012), 253–261
- [11] J. Kosakowska and M. Schmidmeier, *Operations on arc diagrams and degenerations for invariant subspaces of linear operators*, preprint 2012, 37 pp., to appear in Trans. Amer. Math. Soc., [arXiv:1202.2813 \[math.RT\]](#)
- [12] J. Kosakowska and M. Schmidmeier, *Arc diagram varieties*, Contemporary Mathematics series of the AMS **607**, 2014, <http://dx.doi.org/10.1090/conm/607/12088>.
- [13] J. Kosakowska, M. Schmidmeier, H. Thomas, *Two partial orders for Littlewood-Richardson tableaux*, preprint (2014), 13 pp.
- [14] H. Kraft, *„Geometrische Methoden in der Invariantentheorie“*, Aspekte der Mathematic, Vieweg (1984)
- [15] M. A. A. van Leeuwen, *Flag varieties and interpretations of Young tableau algorithms*, J. Algebra **224** (2000) 397–426.
- [16] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1995
- [17] T. Maeda, *A partial order on Littlewood-Richardson tableaux*, J. Algebra **319** (2008) 3621–3652.
- [18] M. Reineke, *Generic extensions and multiplicative bases of quantum groups at $q = 0$* , Represent. Theory **5** (2001), 147–163.
- [19] M. Reineke, *The monoid of families of quiver representations*, Proc. London Math. Soc. **84** (2002), 663–685.
- [20] C. Riedtmann, *Degenerations for representations of quiver with relations*, Ann. Sci. Ec. Norm. Super. **4** (1986), 275–301.
- [21] C. M. Ringel and M. Schmidmeier, *Invariant subspaces of nilpotent linear operators. I*, J. Reine Angew. Math. **614** (2008), 1–52.

- [22] M. Schmidmeier, *The entries in the LR-tableau*, Math. Z. **268** (2011), 211–222.
- [23] G. Zwara, *Degenerations for representations of extended Dynkin quivers*, Comment. Math. Helv. **73** (1998), 71–88.

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